波動率估計及其應用

-以地球科學為例

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Outline

- Volatility Estimation: Corrected Fourier Transform Method
- Local and Stochastic Volatility Estimation
- Risk Management: Value at Risk (VaR) / Conditional Value at Risk (CVaR)
- Stress Test and Backtesting for VaR Estimation

Volatility Estimation Problem

- Assume the diffusion process $u(t)$ follows

$$du(t) = \mu(t)dt + \sigma(t)dW_t,$$

- Task: Given return time series $u(t)$, $0 \leq t \leq T$, estimate the instantaneous volatility $\sigma(t)$ for $0 \leq t \leq T$.

Literature Review

- Approximation by the integrated volatility: Zhang et al. (2005) use quadratic variation.
- Fourier Series Representation: Malliavin and Mancino(2002, 2009) propose a Fourier transform method
- These two methods are nonparametric!

Fourier Transform Method - I

- Compute the Fourier coefficients of du by

$$a_0(du) = \frac{1}{2\pi}\int_0^{2\pi} du(t),$$

$$a_k(du) = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)\, du(t),$$

$$b_k(du) = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)\, du(t).$$

- Then,

$$u(t) = a_0 + \sum_{k=1}^{\infty}\left[-\frac{b_k(du)}{k}\cos(kt) + \frac{a_k(du)}{k}\sin(kt)\right].$$

Fourier Transform Method - II

- Fourier coefficients of variance $\sigma^2(t)$,

$$a_k\left(\sigma^2\right)=\lim_{N\to\infty}\frac{\pi}{2N+1}\sum_{s=-N}^{N-k}\left[a_s^*\left(du\right)a_{s+k}^*\left(du\right)+b_s^*\left(du\right)b_{s+k}^*\left(du\right)\right],$$

$$b_k\left(\sigma^2\right)=\lim_{N\to\infty}\frac{\pi}{2N+1}\sum_{s=-N}^{N-k}\left[a_s^*\left(du\right)b_{s+k}^*\left(du\right)-b_s^*\left(du\right)a_{s+k}^*\left(du\right)\right],$$

where n_0 is any positive integer so that

$$\sigma_N^2\left(t\right)=\sum_{k=0}^{N}\left[a_k\left(\sigma^2\right)\cos\left(kt\right)+a_k\left(\sigma^2\right)\sin\left(kt\right)\right].$$

Fourier Transform Method - III

- Reconstruct the time series variance $\sigma^2(t)$.
- Finally, $\sigma_N^2(t)$ is an approximation of $\sigma^2(t)$ as N approaches infinity, which can be given by classical Fourier-Fejer inversion formula.

$$\sigma^2(t) = \lim_{N \to \infty} \sigma_N^2(t) \quad \text{in prob.}$$

Smoothing

$$\sigma^2(t) = \lim_{N\to\infty} \sum_{k=0}^{N} \varphi(\delta k)\left[a_k(\sigma^2)\cos(kt) + b_k(\sigma^2)\sin(kt)\right],$$

where $\varphi(x) = \dfrac{\sin^2(x)}{x^2}$ is a smooth function and δ is a smoothing parameter.

- Reno (2008) alerts the boundary effect in the Fourier transform method.

A Price Correction Scheme: First Order

- Idea: (Nonlinear) Least Squares Method for first-order correction

$$r_t \approx \sigma_t \delta_t \varepsilon_t$$

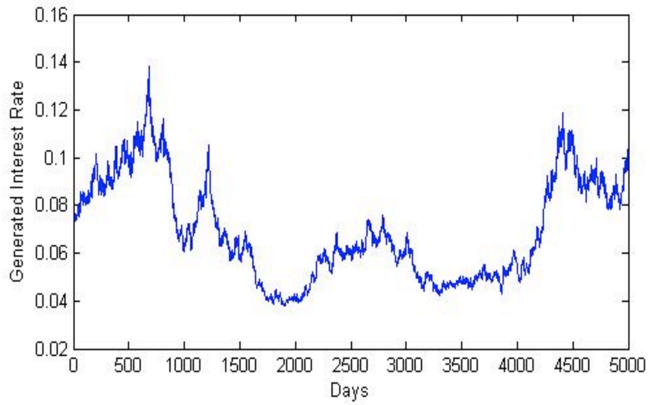
$$\approx \exp\left(\left(a + b\hat{Y}_t\right)/2\right)\delta_t \varepsilon_t .$$

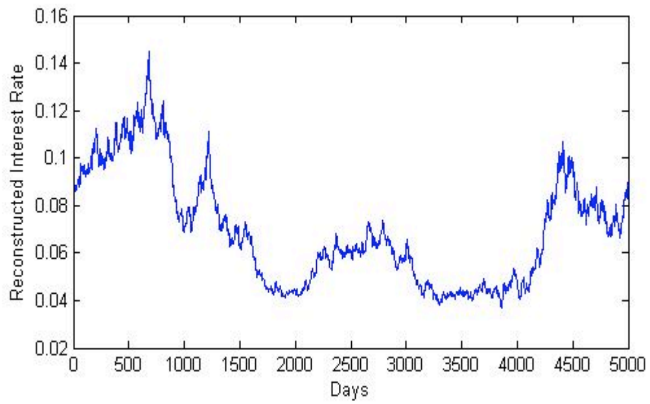
- Then by MLE to regress out a and b

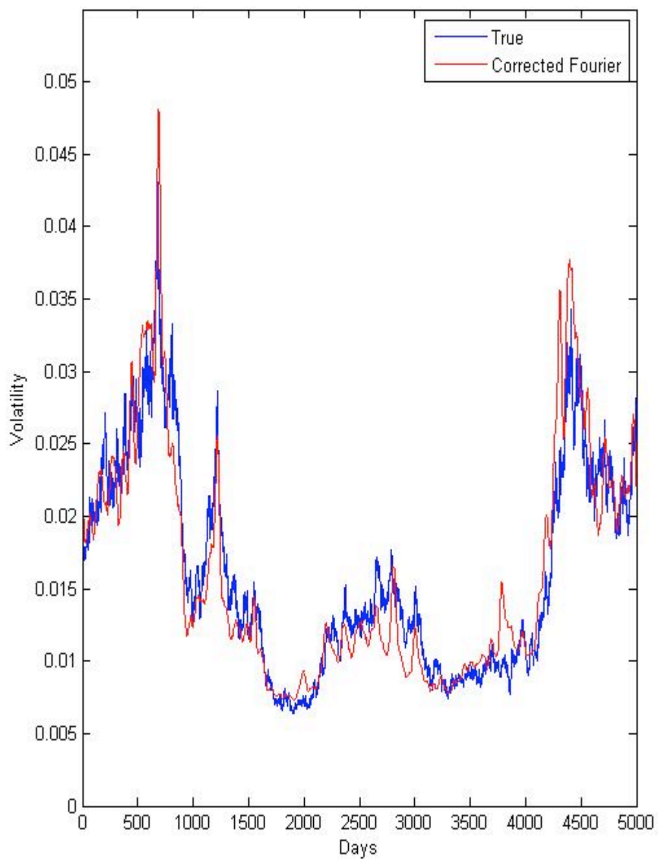
$$\ln\left(\frac{r_t}{\delta_t}\right)^2 = a + b\hat{Y}_t + \ln \varepsilon_t^2 .$$

Simulation Study – Local Volatility

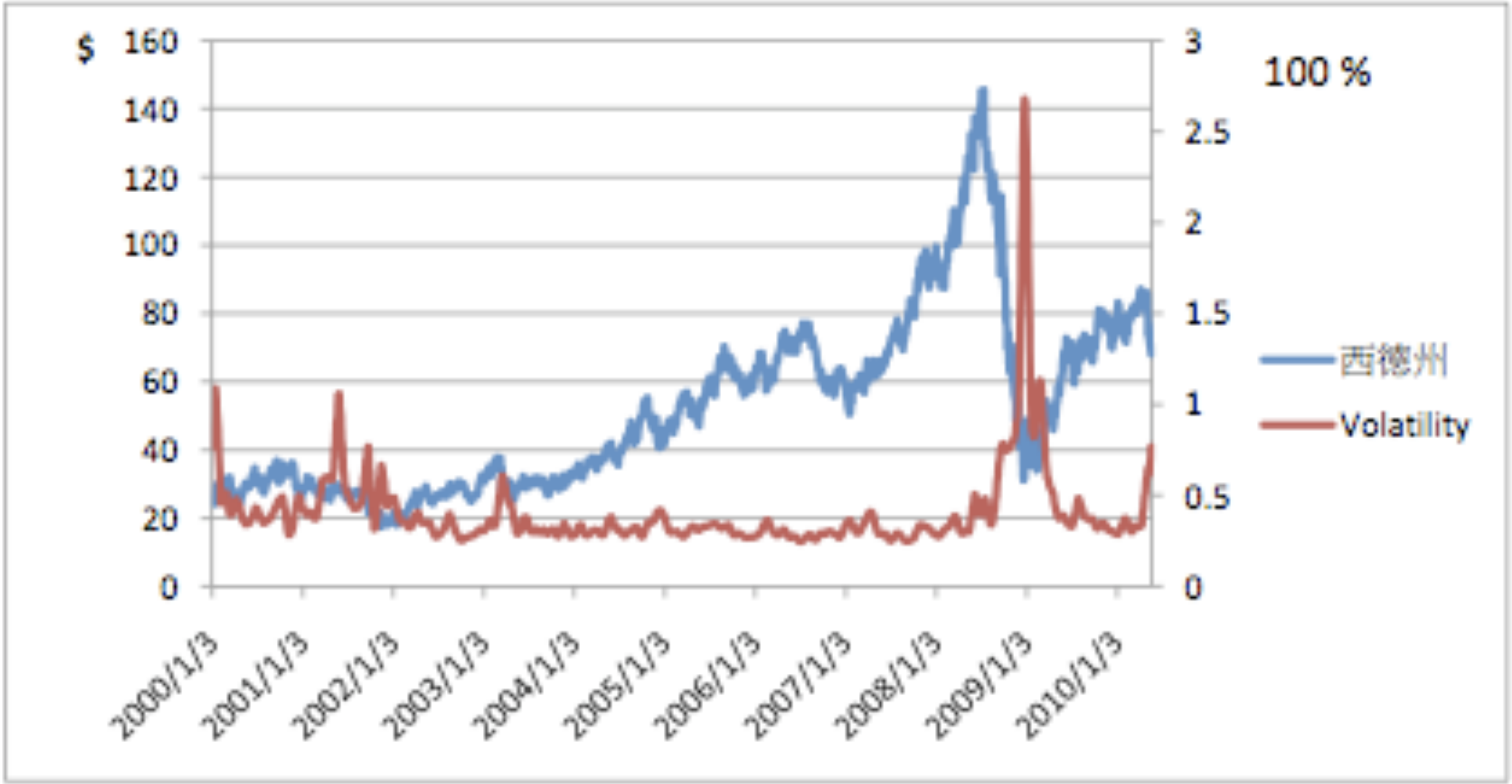
$$\sigma_t = \beta r_t^{\gamma}$$



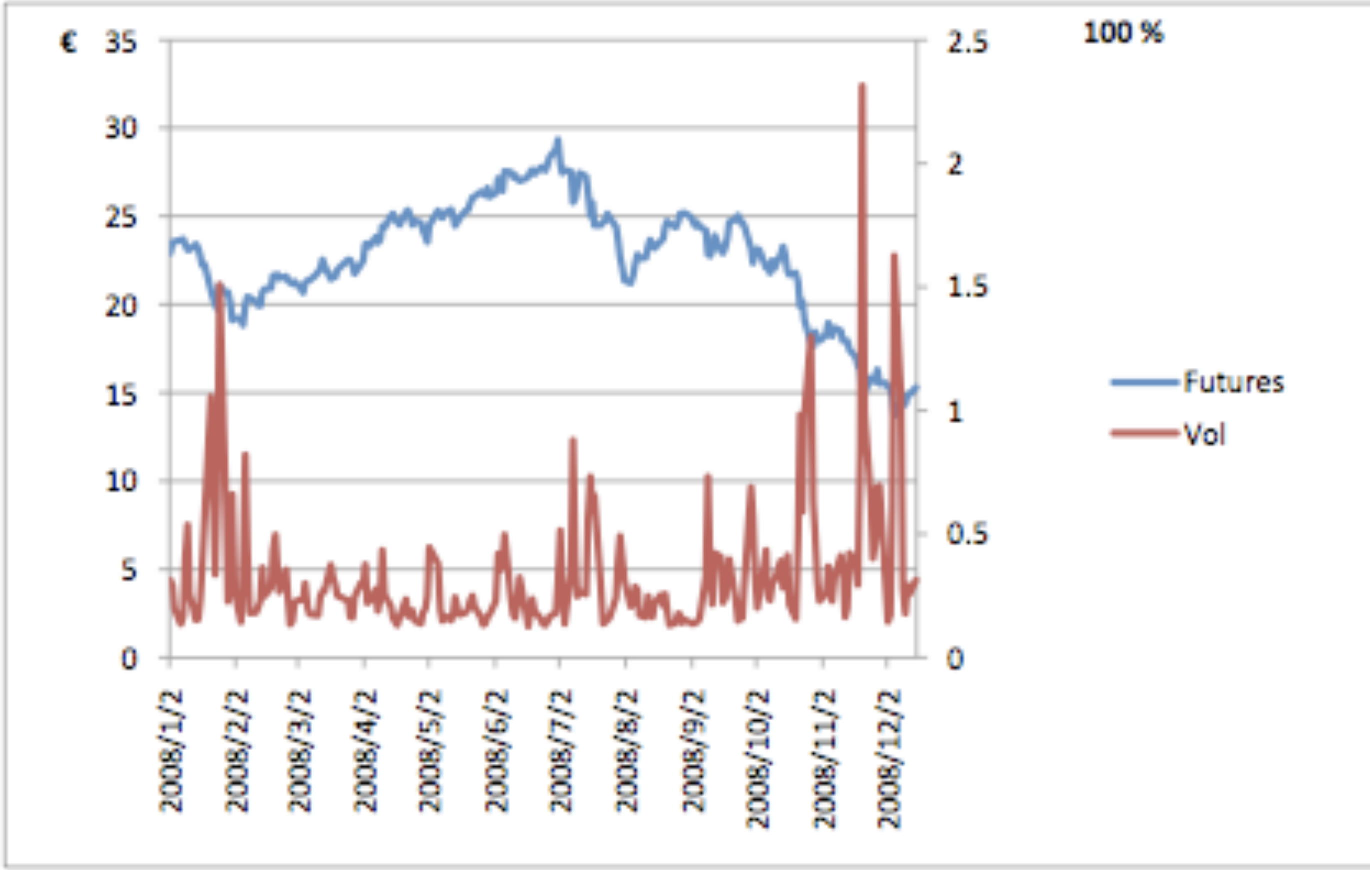




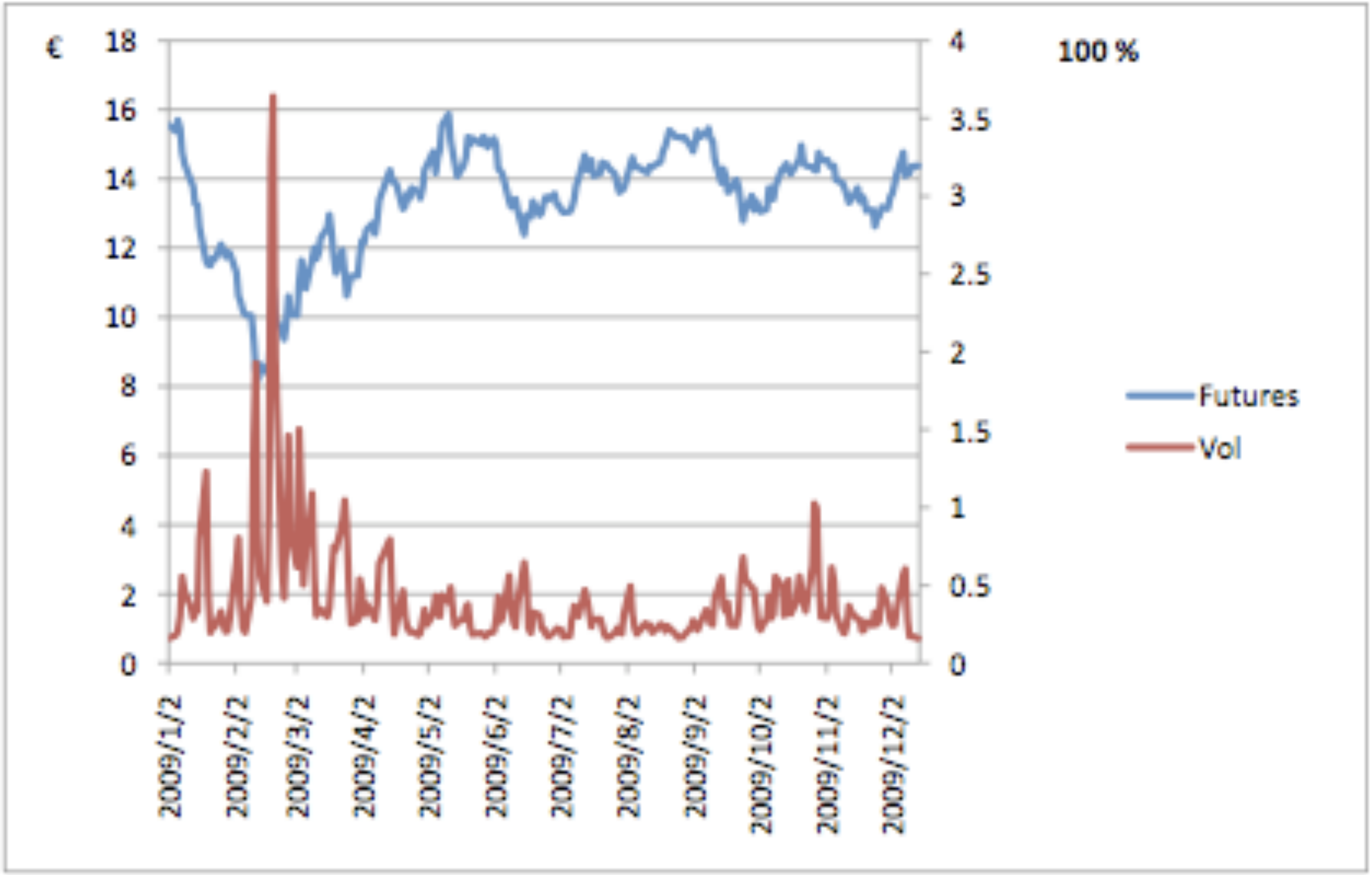
Oil Price & Volatility



Carbon Futures Price & Vol – I



Carbon Futures Price & Vol – II



Stochastic Volatility Model

$$d\,u(t) = \mu(t)\,dt + \sigma(t)\,dW_t$$

$$\sigma(t) = f(Y_t)$$

$$dY_t = \alpha(m - Y_t)\,dt + \beta\,dZ_t$$

- Note that the model estimation is NOT a standard filtering problem.

Stochastic Volatility Estimation Methods: Overview

- Broto and Ruiz (2004) : method of moments, generalized method of moments, maximum likelihood estimators, quasi maximum likelihood, etc.

- Yu (2010) - simulation-based estimation methods: simulated maximum likelihood, simulated generalized method of moments, efficient method of moments, indirect inference, Markov chain Monte Carlo , etc.

A New Approach: Corrected Fourier Method with MLE

- Once the volatility process $\sigma(t) = f(Y_t)$ is estimated, one can use the state-space method or MLE for stochastic volatility model estimation.

- For Example, assuming that the driving volatility process is governed by the Ornstein-Uhlenbeck process,

$$dY_t = \alpha\left(m - Y_t\right)dt + \beta\, dW_t.$$

Likelihood Function

- For a given set of observations $(Y_1, Y_2, \ldots, Y_N)$ the likelihood function is

$$L(\alpha, \beta, m) = \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi\beta^2\Delta_t}} \exp\left\{-\frac{1}{2\beta^2\Delta_t}\left[Y_{t+1} - \left(\alpha m \Delta_t + (1-\alpha\Delta_t)Y_t\right)\right]^2\right\},$$

where Δ_t denotes the length of discretized time interval.

ML Estimators

$$\hat{\alpha} = \frac{1}{\Delta_t}\left[1-\frac{\left(\sum_{t=2}^{N} Y_t\right)\left(\sum_{t=1}^{N-1} Y_t\right)-(N-1)\left(\sum_{t=1}^{N-1} Y_t Y_{t+1}\right)}{\left(\sum_{t=1}^{N-1} Y_t\right)^2-(N-1)\left(\sum_{t=1}^{N-1} Y_t^2\right)}\right],$$

$$\hat{\beta}=\sqrt{\frac{1}{N\Delta_t}\sum_{t=1}^{N-1}\left[Y_{t+1}-\left(\alpha m\Delta_t+\left(1-\alpha\Delta_t\right)Y_t\right)\right]^2},$$

$$\hat{\mathrm{m}}=\frac{-1}{\hat{\alpha}\Delta_t}\left[\frac{\left(\sum_{t=2}^{N} Y_t\right)\left(\sum_{t=1}^{N-1} Y_t^2\right)-\left(\sum_{t=1}^{N-1} Y_t\right)\left(\sum_{t=1}^{N-1} Y_t Y_{t+1}\right)}{\left(\sum_{t=1}^{N-1} Y_t\right)^2-(N-1)\left(\sum_{t=1}^{N-1} Y_t^2\right)}\right],$$

Simulation Study – Stochastic Vol

- Let the stochastic volatility model is

$$\begin{cases} dS_t = \mu S_t dt + \exp\left(Y_t/2\right) S_t dW_{1t}, \\ dY_t = \alpha\left(m - Y_t\right)dt + \beta dW_{2t}. \end{cases}$$

- Set model parameters as follows:

$$\mu = 0.01,\ S_0 = 50, Y_0 = -2,\ m = -2,\ \alpha=5,\ \beta = 1,$$

with the discretization length $\Delta_t = 1/5000$

Then we generate volatility series $\sigma_t = \exp\left(Y_t/2\right)$ and asset price series S_t .

Simulation Study(cont.)

- Two criteria are used for performance comparison: Mean squared errors (MSE) and Maximum absolute errors (MAE).
- Comparison results are shown below:

	Fourier method	Corrected Fourier method
Mean squared error	0.0324	0.0025
Maximum absolute error	0.3504	0.1563

Up to now…

- Develop a corrected Fourier transform method for volatility estimation
- Combine this method with MLE for stochastic volatility model estimation
- Next, we discuss some practical applications in risk management

Value at Risk

- Let $r(t)$ be an asset return at time t. Its $\alpha \times 100\%$ VaR, denoted by VaR_α , is defined by the $(1-\alpha)\times 100\%$ percentile of $r(t)$. That is,

$$P\left(r(t) \le VaR_\alpha\right) = 1-\alpha$$

Aspects about Risk Measure

- Mathematically, it is not a coherent risk measure* because it doesn't satisfy the risk diversification principal. Instead, CVaR does!

- Practically, VaR is commonly required by financial regulations (since Basel Ⅱ Accord).

* Artzner P., F. Delbaen, J.-M. Eber, and D. Heath (1999).

Estimation of VaR

- Riskmetrics: normal assumption under EWMA model.
- Historical Simulation: generate scenarios
- Model Dependent Approach: Discrete-Time Model vs. Continuous-Time Model

Estimate Extreme Probability

- Given a Markovian dynamic model of an asset price S_t , its return process is $r_T = \ln\left(S_T / S_0\right)$.

- Given a loss threshold D, the extreme probability is defined by

$$P\left(0, S_0; D\right) = E\left[I\left(r_T \le D\right) \middle| S_0\right].$$

Note: solve VaR_α from $P\left(0, S_0; VaR_\alpha\right) = 1 - \alpha.$

- $CVaR = E\left[r_T \middle| r_T \le VaR_\alpha\right].$ (Expected Shortfall)

Importance Sampling for Scaled Stochastic Volatility Model

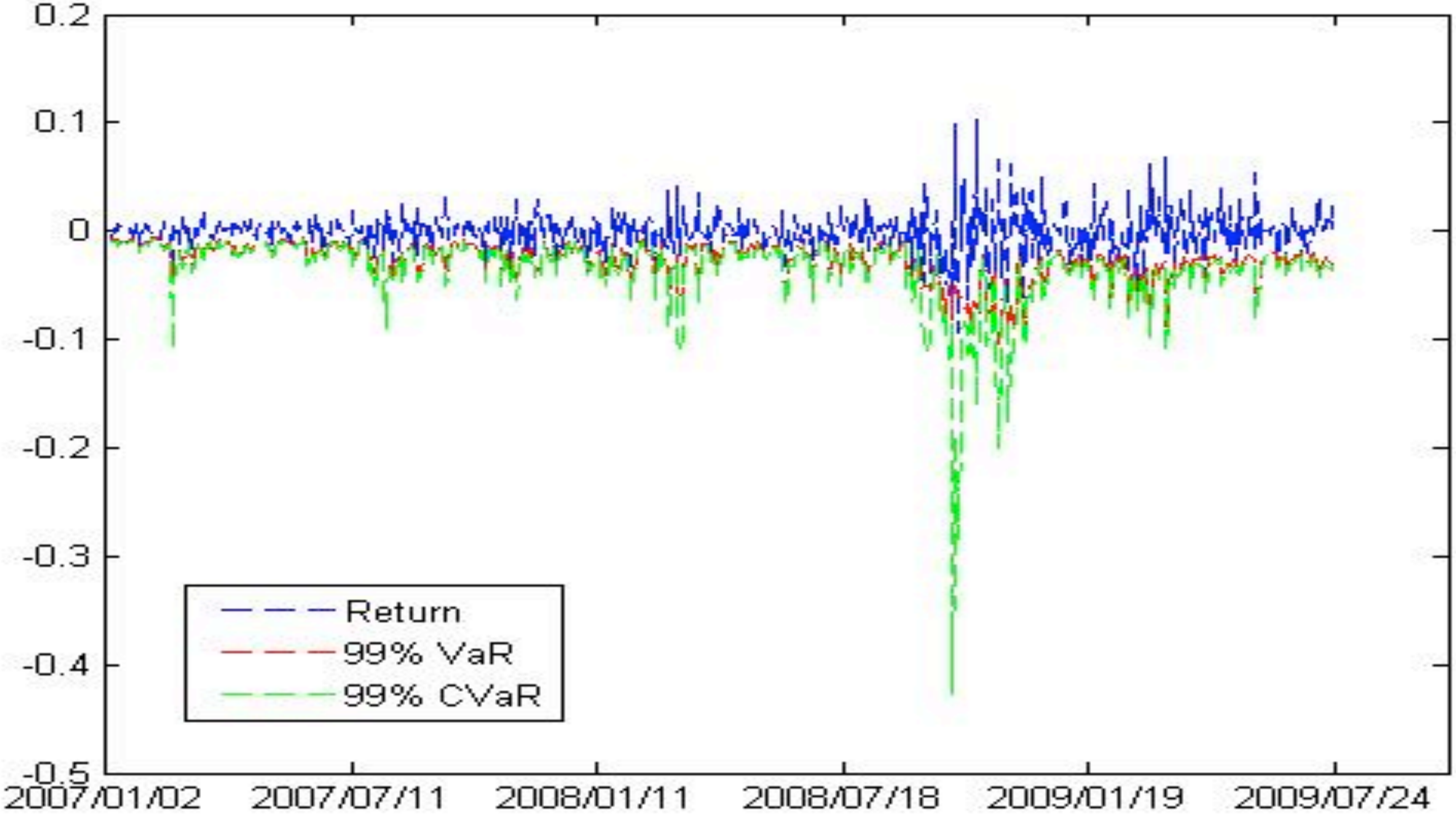
- E.g. $\alpha = \frac{1}{\varepsilon}, \beta = \sqrt{\frac{2\nu}{\varepsilon}}.$

- By an averaging property, we develop an efficient importance sampling method for VaR estimation.

- CVaR can also be easily estimated.

VaR/CVaR Estimation: S&P 500

- Data sample period: 2005.01.03-2009.07.24



Stress Test and Backtesting – S&P 500

Data sample period: 2005.01.03-2009.07.24

Historical Simulation			
Significance	1%	Significance	5%
LRuc	Reject VaR Model	LRuc	Reject VaR Model
LRind	Reject VaR Model	LRind	**Don't Reject VaR Model**
LRcc	Reject VaR Model	LRcc	Reject VaR Model
RiskMetrics			
Significance	1%	Significance	5%
LRuc	Reject VaR Model	LRuc	Reject VaR Model
LRind	**Don't Reject VaR Model**	LRind	**Don't Reject VaR Model**
LRcc	Reject VaR Model	LRcc	Reject VaR Model
GARCH(1,1)			
Significance	1%	Significance	5%
LRuc	Reject VaR Model	LRuc	Reject VaR Model
LRind	**Don't Reject VaR Model**	LRind	Reject VaR Model
LRcc	Reject VaR Model	LRcc	Reject VaR Model
SV			
Significance	1%	Significance	5%
LRuc	**Don't Reject VaR Model**	LRuc	Reject VaR Model
LRind	**Don't Reject VaR Model**	LRind	**Don't Reject VaR Model**
LRcc	**Don't Reject VaR Model**	LRcc	Reject VaR Model

Backtesting Outcomes of JPY/USD VaR Estimate

- Data sample period: 1998.01.05-2009.07.24

Historical Simulation			
Significance	1%	Significance	5%
LRuc	**Don't Reject VaR Model**	LRuc	**Don't Reject VaR Model**
LRind	**Don't Reject VaR Model**	LRind	Reject VaR Model
LRcc	**Don't Reject VaR Model**	LRcc	Reject VaR Model
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LRcc	Reject VaR Model	LRcc	**Don't Reject VaR Model**
SV			
Significance	1%	Significance	5%
LRuc	**Don't Reject VaR Model**	LRuc	**Don't Reject VaR Model**
LRind	**Don't Reject VaR Model**	LRind	**Don't Reject VaR Model**
LRcc	**Don't Reject VaR Model**	LRcc	**Don't Reject VaR Model**

Conclusion

- Bias Reduction - remedy volatility boundary deficit of Fourier transform method by a price correction scheme

- Variance Reduction - efficient importance sampling to estimate VaR/CVaR under Stochastic Volatility models.

- Backtesting - VaR backtesting for FX and equity data, SV model outperforms.

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Thank You for Your Patient!